

An infinitesimal approach to a conjecture of Eisenbud and Harris

JÜRGEN RATHMANN

1. INTRODUCTION

Eisenbud and Harris conjectured that an algebraic curve of high genus lies on a surface of low degree (which they proved for curves of very large degree). They observed constraints on the Hilbert function of a general hyperplane section Γ , which imply that Γ lies on a curve C of low degree. We investigate this situation under deformation. Given a set of sufficiently many points $\Gamma \subset C$ with C linearly normal, we show that for every deformation of $\Gamma \subset \mathbf{P}^r$ with constant $h_\Gamma(2)$ the curve C deforms along with Γ .

Let X be a reduced irreducible nondegenerate curve of degree n and genus g in \mathbf{P}^{r+1} . Castelnuovo showed in the late 19th century that the genus is bounded by a quadratic polynomial $\pi_0(n, r)$ in n with leading term $\frac{n^2}{2r}$. He also showed that curves of maximal genus must lie on a unique surface S of minimal degree r , at least as long as $n \geq 2r + 3$.

Castelnuovo's approach was extended by Eisenbud and Harris in 1982 who defined functions $\pi_\alpha(n, r) = \frac{n^2}{2(r+\alpha)} + O(n)$ ($1 \leq \alpha \leq r$) and $n_0(r)$ and showed that any curve X with degree $n \geq n_0$ and genus $g > \pi_\alpha(n, r)$ must lie on a surface of degree $\leq r - 1 + \alpha$. [7, 3.22]. Here $n_0(r) = 2^{r+2}$ for $r \geq 7$ (slightly worse for $r < 7$).

They further conjectured that the restriction on n could be lowered:

Conjecture 1.1 (Eisenbud-Harris). *Let $X \subset \mathbf{P}^{r+1}$ be a nondegenerate reduced irreducible curve of genus g and degree $n \geq 2r + 3 + 2\alpha$. If $g > \pi_\alpha(n, r)$, then X lies on a surface of degree at most $r - 1 + \alpha$.*

They proved the conjecture for $\alpha = 1$ [7, 3.15]. The next case $\alpha = 2$ ($r \geq 7$) was established by Petrakiev in 2008 [9].

The proofs proceed by a careful analysis of the Hilbert function $h_\Gamma(l)$ of a general hyperplane section $\Gamma = X \cap H$. Here $h_\Gamma(l)$ is the rank of the l -th graded piece of the homogeneous coordinate ring of Γ .

Castelnuovo obtains his bound from showing that $h_\Gamma(l) \geq \min(n, lr + 1)$. On the other hand, if $n \geq 2r + 3$ and $h_\Gamma(2) = 2r + 1$, then the quadrics containing Γ intersect in a unique rational normal curve of degree r in H . The surface S of minimal degree is obtained as the intersection of all quadrics which contain X .

Eisenbud and Harris' proof for $\alpha = 1$ uses that $h_\Gamma(2) = 2r + 2$ for curves with $n \geq 2r + 5$ and $g > \pi_1(n, r)$. They show that the quadrics through Γ intersect in an elliptic normal curve.

These approaches suggest the following supporting conjecture which (except for part (ii)) has been noted and studied by Reid [10], and later by Petrakiev [9]:

Conjecture 1.2. *Let $\Gamma \subset \mathbf{P}^r$ be a general hyperplane section of a nondegenerate reduced irreducible curve of degree n . Assume that Γ imposes $2r + 1 + \alpha$ conditions on quadrics for some $0 \leq \alpha \leq r - 2$. If one of the following holds*

- (i) $\alpha \leq r - 3$ and $n \geq 2r + 3 + 2\alpha$,
- (ii) $\alpha = r - 2$ and $n \geq 4r + 1 = 2r + 5 + 2\alpha$,

then Γ lies on a reduced irreducible curve C of degree at most $r + \alpha$. C is a component of the intersection of the quadrics containing Γ .

For fixed r , the numerical restrictions on n and α cannot be improved [10],[3].

In addition to the cases $\alpha = 0, 1$ mentioned above, this conjecture is known for $\alpha = 2$, $r \geq 5$ and $\alpha = 3$, $r \geq 7$ [9, prop. 4.3 and 4.4]. For arbitrary $\alpha \leq r - 3$, it is known, if Γ lies on a two-dimensional rational normal scroll [5], [10].

In this note, we try to collect further evidence by investigating the following infinitesimal version of (1.2):

Question 1.3. *Suppose Γ is a set of n points in \mathbf{P}^r satisfying both the assumptions and the conclusion of (1.2). Is every deformation of Γ induced by a deformation of C ?*

Our setup deals with the simplest case, namely that of linearly normal smooth curves.

To describe our results, let $0 \leq \alpha \leq r - 2$ and consider the open subset of the Hilbert scheme of curves of genus $g = \alpha$ and degree $d = g + r$ corresponding to smooth irreducible curves embedded by a complete linear system. Using the universal family, we can construct a variety $H(n, \alpha)$ parametrizing subschemes of such curves consisting of pairwise distinct points. $H(n, \alpha)$ is smooth and irreducible.

Further, let $P(n)$ be the open subset of the Hilbert scheme of length n -subschemes of \mathbf{P}^r corresponding to subschemes Γ consisting of pairwise distinct points, and consider the canonical restriction map

$$H^0 \mathcal{O}_{\mathbf{P}^r}(2) \xrightarrow{M} H^0 \mathcal{O}_{\Gamma}(2)$$

on $P(n)$.

$$P(n, \alpha) = \{(x_1, \dots, x_n) \in P(n) : \text{rk}(M) \leq 2r + \alpha + 1\}$$

is a determinantal variety.

The embedding of C in \mathbf{P}^r yields a map $F(n, \alpha): H(n, \alpha) \rightarrow P(n, \alpha)$.

Theorem 1.4. *If $0 \leq \alpha \leq r - 3$ and $n \geq 2r + 3 + 2\alpha$ or $\alpha = r - 2$ and $n \geq 4r + 1$, then $F(n, \alpha)$ maps $H(n, \alpha)$ isomorphically on a dense open subset of a component of $P(n, \alpha)$.*

Theorem 1.5. *If $0 \leq \alpha \leq r - 2$ and $n \geq 2r + \alpha + 5$, and C is a fixed curve, then $F(n, \alpha)$ is locally an isomorphism in a general subscheme Γ of C .*

Corollary 1.6. *If $0 \leq \alpha \leq r - 2$ and $n \geq 2r + \alpha + 5$, then $P(n, \alpha)$ has an irreducible component whose general point lies in the image of $F(n, \alpha)$.*

The bounds in (1.4) correspond to those predicted by (1.2). They are sharp, see (3.4).

Our proofs proceed by analyzing the corresponding map on the Zariski tangent spaces. Each statement is equivalent to the surjectivity of the canonical composite map

$$H^0 \mathcal{I}_C(2) \otimes H^0 B \longrightarrow H^0(N_{C|\mathbf{P}}^\vee(2)) \otimes H^0 B \longrightarrow H^0(N_{C|\mathbf{P}}^\vee(2) \otimes B).$$

for $B = \omega_C \otimes \mathcal{O}_C(\Gamma - 2)$.

Writing

$$P_L = \text{Ker}((H^0 \mathcal{I}_C(2)) \otimes \mathcal{O}_C \longrightarrow N_{C|\mathbf{P}}^\vee(2)),$$

this question reduces to showing that $H^1(P_L \otimes B) = 0$.

As P_L is a quotient of $(pr_1)_*(pr_2^*L \otimes pr_3^*L \otimes \mathcal{O}(-\Delta_{1,2} - \Delta_{1,3} - \Delta_{2,3}))$ ($pr_i: C \times C \times C \rightarrow C$ the canonical projections), we are led to study conditions on line bundles L_1, L_2 and L_3 which guarantee the vanishing of

$$H^1(pr_1^*L_1 \otimes pr_2^*L_2 \otimes pr_3^*L_3 \otimes \mathcal{O}(-\Delta_{1,2} - \Delta_{1,3} - \Delta_{2,3})).$$

The latter question can be resolved by a suitable filtration of a direct image, similarly to the approach in [8].

Interestingly, our proof does not require the points of Γ to be in linearly general position. If the curve C has degree $2g + 2$, it cannot have a trisecant line, but may have a foursecant plane. A 5-secant plane is not possible, but there could be a 5-secant 3-plane; for hyperelliptic C , there could even be a 6-secant-3-plane.

The proof even works for an arbitrary divisor on C without assuming that its support consists of pairwise distinct points.

In the last section, we briefly comment on our assumption of linear normality of C and give two applications of our methods, on the question of generation of ideal sheaves by quadrics and on the cohomology of the square of the ideal sheaf.

These results were obtained in 1990–91 at UCLA, but remained unpublished when I did not continue my academic career. As I recently found that the techniques can be applied to study syzygies of curves, I decided to publish the results, after updating my notes with more recent developments.

I am grateful to Rob Lazarsfeld for many helpful discussions and encouragement. I am also indebted to Aaron Bertram who first asked the questions answered by (3.2) and (4.2) (they were communicated to me by Lazarsfeld) which turned out to be pivotal for this paper. Furthermore, I profited from discussions with Mark Green and Christoph Rippel.

NOTATIONS AND CONVENTIONS

We work over an algebraically closed field K of arbitrary characteristic.

Given a curve C , we will write pr_1, pr_2 for the projections on $C \times C$, and Δ for the diagonal. On $C \times C$, we define sheaves associated with a line bundle L as follows: $M_L = (pr_1)_*(pr_2^*L \otimes \mathcal{O}_{C \times C}(-\Delta))$, $R_L = (pr_1)_*(pr_2^*L \otimes \mathcal{O}_{C \times C}(-2\Delta))$.

If L is very ample and $C \rightarrow \mathbf{P}^r$ is the embedding by $H^0 L$, then $M_L = \Omega_{\mathbf{P}}^1(1)|_C$, $R_L = L \otimes N_{C/\mathbf{P}}^\vee$.

pr_i also denotes the projections from $C^3 = C \times C \times C$ onto the i -th component, and $pr_{1,2}$ denotes the projection on the first two components.

$\Delta_{i,j}$ is the diagonal $\{(x_1, x_2, x_3) | x_i = x_j\} \subset C \times C \times C$.

For a line bundle L , let $M_{2,L} = (pr_{1,2})_*(pr_3^* L \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,3} - \Delta_{2,3}))$. If L is very ample, then $M_{2,L}$ is a vector bundle on $C \times C$.

For very ample L so that the ideal of C in the embedding by $H^0 L$ is generated by quadrics, we write $P_L = \text{Ker}(H^0 \mathcal{I}_C(2) \otimes \mathcal{O}_C \rightarrow N_{C/\mathbf{P}}^\vee(2))$.

2. DEFORMING SETS OF POINTS

Let C be a curve of genus g , embedded in \mathbf{P}^r by a complete linear system of degree $d = g + r$ for some $r \geq g + 2$. Then $\deg(C) \geq 2g + 2$ and C satisfies (N_1) , i.e., the homogeneous ideal of C is generated by quadrics. Note that C imposes $h^0 \mathcal{O}_C(2) = 2d + 1 - g = 2r + 1 + g$ conditions on quadrics.

In the situation of (1.4), C is uniquely determined as the intersection of the quadrics containing Γ , since Γ consists of $n \geq 2r + 3 + 2g = 2d + 3 > 2d$ points.

In the situation of (1.5), the same holds for general Γ , because the residual divisor for a quadric through n general points on C consists of a section of a line bundle of degree $2d - n \leq g - 5 < g$, and the general such line bundle has only the zero section.

As $F(n, g)$ is injective at Γ under our assumptions, it suffices to show that the induced map on Zariski tangent spaces is an isomorphism. We will use the following local criterion:

Proposition 2.1. *Let C be a curve of genus $g = \alpha$, embedded into \mathbf{P}^r by a non-special complete linear system. Assume that C satisfies (N_1) . Let $\Gamma = \{x_1, \dots, x_n\} \subset C$ with $H^0 \mathcal{O}_C(2 - \Gamma) = 0$ and set $B = \omega_C \otimes \mathcal{O}_C(\Gamma - 2)$. The following conditions are equivalent:*

- (i) *$F(n, g)$ induces a bijective map on Zariski tangent spaces in Γ .*
- (ii) *The multiplication map $H^0(\mathcal{I}_C(2)) \otimes H^0 B \rightarrow H^0(N^\vee(2) \otimes B)$ is surjective.*

Proof. The Zariski tangent space of $H(n, g)$ has a filtration with two components, namely $H^0 N_{\Gamma/C}$ representing deformations of Γ in C , and $H^0 N_{C/\mathbf{P}}$ representing deformations of C in \mathbf{P}^r .

$P(n, g)$ is a determinantal variety, and its Zariski tangent space at Γ is obtained by pullback from the generic determinantal variety $M_k = \{A \in K^{m \times n} : \text{rk}(A) \leq k\}$ (see [2, II.2]). The Zariski tangent space at $A \in M_k - M_{k-1}$ is $T_A(M_k) = \{B \in K^{m \times n} : B \cdot \text{Ker}(A) \subset \text{Im}(A)\}$.

In our case, $T_\Gamma(P(n)) = H^0 N_{\Gamma/\mathbf{P}}$, $\text{Ker}(M) = H^0 \mathcal{I}_\Gamma(2)$, $\text{Im}(M) = \text{Im}(H^0 \mathcal{O}_{\mathbf{P}^r}(2) \rightarrow H^0 \mathcal{O}_\Gamma(2))$. Therefore

$$T_\Gamma(P(n, g)) = \text{Ker}(H^0 N_{\Gamma/\mathbf{P}} \rightarrow \text{Hom}(H^0 \mathcal{I}_\Gamma(2), H^1 \mathcal{I}_\Gamma(2))).$$

Now consider the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & H^0 N_{\Gamma/C} & & & & \\
 & & \downarrow & & & & \\
 & & H^0 N_{\Gamma/\mathbf{P}} & \longrightarrow & \mathrm{Hom}(H^0 \mathcal{I}_{\Gamma}(2), H^1 \mathcal{I}_{\Gamma}(2)) & & \\
 & & \downarrow & & & & \\
 H^0 N_{C/\mathbf{P}} & \longrightarrow & H^0(N_{C/\mathbf{P}}|_{\Gamma}) & \longrightarrow & H^1(N_{C/\mathbf{P}} \otimes \mathcal{O}_C(-\Gamma)) & \longrightarrow & H^1 N_{C/\mathbf{P}} \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

The first map in the bottom line is injective: $H^0(N_{C/\mathbf{P}} \otimes \mathcal{O}_C(-\Gamma)) = 0$, because Γ determines C , hence no non-trivial deformation of C can fix Γ .

$H^1 N_{C/\mathbf{P}} = 0$, because C is embedded by a nonspecial linear system, hence C is unobstructed.

$H^0 N_{\Gamma/C}$ classifies deformations of Γ inside C . This vector space lies in the Zariski tangent space of $P(n, g)$ at Γ , and we obtain a map

$$H^0(N_{C/\mathbf{P}}|_{\Gamma}) \longrightarrow \mathrm{Hom}(H^0 \mathcal{I}_{\Gamma}(2), H^1 \mathcal{I}_{\Gamma}(2)).$$

$H^0 N_{C/\mathbf{P}}$ classifies deformations of C in \mathbf{P}^r . As the deformed curve lies on the same number of quadrics, this vector space also lies in the Zariski tangent space of $P(n, g)$ at Γ .

We obtain: C deforms with Γ if and only if the map

$$f : H^1(N_{C/\mathbf{P}} \otimes \mathcal{O}_C(-\Gamma)) \longrightarrow \mathrm{Hom}(H^0 \mathcal{I}_{\Gamma}(2), H^1 \mathcal{I}_{\Gamma}(2))$$

is injective.

We identify $H^0 \mathcal{I}_{\Gamma}(2) = H^0 \mathcal{I}_C(2)$ and $H^1 \mathcal{I}_{\Gamma}(2) = H^1 \mathcal{I}_{\Gamma, C}(2) = H^1 \mathcal{O}_C(2 - \Gamma)$, and dualize f :

$$\begin{array}{ccc}
 \mathrm{Hom}(H^1 \mathcal{O}_C(2 - \Gamma), H^0 \mathcal{I}_C(2)) & \longrightarrow & H^1(N_{C/\mathbf{P}} \otimes \mathcal{O}_C(-\Gamma))^\vee \\
 \downarrow \cong & & \\
 H^1 \mathcal{O}_C(2 - \Gamma)^\vee \otimes H^0 \mathcal{I}_C(2) & &
 \end{array}$$

Applying Serre duality, we find a map

$$f^\vee : H^0(\omega_C \otimes \mathcal{O}_C(\Gamma - 2)) \otimes H^0 \mathcal{I}_C(2) \longrightarrow H^0(N_{C/\mathbf{P}}^\vee \otimes \omega_C \otimes \mathcal{O}_C(\Gamma))$$

which is the composition

$$\begin{aligned} H^0(\omega_C \otimes \mathcal{O}_C(\Gamma - 2)) \otimes H^0 \mathcal{I}_C(2) &\longrightarrow H^0(\omega_C \otimes \mathcal{O}_C(\Gamma - 2)) \otimes H^0 N_{C/\mathbf{P}}^\vee(2) \\ &\longrightarrow H^0(N_{C/\mathbf{P}}^\vee \otimes \omega_C \otimes \mathcal{O}_C(\Gamma)). \end{aligned}$$

C deforms with Γ if and only if f^\vee is surjective. \square

Proposition 2.2. *Let L be a very ample line bundle on a curve C and assume that the ideal sheaf of C is generated by quadrics. There is an exact sequence*

$$\begin{aligned} 0 \longrightarrow \wedge^2 M_L \longrightarrow (pr_1)_*(pr_2^* L \otimes pr_3^* L \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,2} - \Delta_{1,3} - \Delta_{2,3})) \\ \longrightarrow (H^0 \mathcal{I}_C(2)) \otimes \mathcal{O}_C \longrightarrow N_{C/\mathbf{P}}^\vee(2) \longrightarrow 0. \end{aligned}$$

Proof. On $C \times C \times C$, we have the following diagram with exact rows and columns

$$(1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}(-\Delta_{1,3} - \Delta_{2,3}) & \longrightarrow & \mathcal{O}(-\Delta_{2,3}) & \longrightarrow & \mathcal{O}_{\Delta_{1,3}}(-\Delta) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}(-\Delta_{1,3}) & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}_{\Delta_{1,3}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{\Delta_{2,3}}(-\Delta) & \longrightarrow & \mathcal{O}_{\Delta_{2,3}} & \longrightarrow & \mathcal{O}_\Delta \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Tensoring with $pr_3^* L$ and applying $(pr_{1,2})_*$, we obtain a diagram on $C \times C$

$$(2) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_{2,L} & \longrightarrow & pr_2^* M_L & \longrightarrow & pr_1^* L \otimes \mathcal{O}(-\Delta) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & pr_1^* M_L & \longrightarrow & H^0 L \otimes \mathcal{O} & \longrightarrow & pr_1^* L \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & pr_2^* L \otimes \mathcal{O}(-\Delta) & \longrightarrow & pr_2^* L & \longrightarrow & L \otimes \mathcal{O}_\Delta \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Applying $(pr_1)_*$ to the second exterior power of the top horizontal sequence, we find that $(pr_1)_*(\wedge^2 M_{2,L})$ injects into $(pr_1)_*(\wedge^2 pr_2^* M_L) = (H^0(\wedge^2 M_L)) \otimes \mathcal{O}_C$. However, $\wedge^2 M_L$ has no global sections, hence $(pr_1)_*(\wedge^2 M_{2,L}) = 0$.

Considering next $(pr_1)_*$ of the left vertical sequence, we find a monomorphism $\wedge^2 M_L \longrightarrow (pr_1)_*(M_{2,L} \otimes pr_2^* L \otimes \mathcal{O}(-\Delta))$.

We now tensor diagram (2) with $pr_2^* L \otimes \mathcal{O}(-\Delta)$, apply $(pr_1)_*$ and divide the four direct image sheaves in the top left square by the image of $\wedge^2 M_L$.

The middle horizontal row of the resulting diagram appears as the bottom row of the following diagram

$$\begin{array}{ccccccc}
 & \wedge^2 M_L & \xlongequal{\quad} & \wedge^2 M_L & & & \\
 & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & M_L \otimes M_L & \longrightarrow & M_L \otimes H^0 L & \longrightarrow & M_L \otimes L \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & S^2 M_L & \longrightarrow & \mathcal{F} & \longrightarrow & M_L \otimes L \longrightarrow 0
 \end{array}$$

We find that \mathcal{F} (defined here as a push-down) is part of the filtration of $S^2 H^0 L \otimes \mathcal{O}_C$, induced by taking second symmetric powers of

$$0 \longrightarrow M_L \longrightarrow H^0 L \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0.$$

For the full filtration, see the middle row of the next commutative diagram

$$\begin{array}{ccccccc}
 & H^0 \mathcal{I}_C(2) \otimes \mathcal{O}_C & \xlongequal{\quad} & H^0 \mathcal{I}_C(2) \otimes \mathcal{O}_C & & & \\
 & \downarrow & & \downarrow & & & \\
 (3) \quad 0 & \longrightarrow & \mathcal{F} & \longrightarrow & S^2 H^0 L \otimes \mathcal{O}_C & \longrightarrow & L^2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & M_{L^2} & \longrightarrow & H^0 L^2 \otimes \mathcal{O}_C & \longrightarrow & L^2 \longrightarrow 0
 \end{array}$$

The left vertical sequence thus is the middle vertical sequence of the desired result

diagram, and we obtain

$$(4) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P_L & \longrightarrow & H^0 \mathcal{I}_C(2) \otimes \mathcal{O}_C & \longrightarrow & R_L \otimes L \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S^2 M_L & \longrightarrow & \mathcal{F} & \longrightarrow & M_L \otimes L \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R_{L^2} & \longrightarrow & M_{L^2} & \longrightarrow & \omega_C \otimes L^2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The sequence from our proposition arises from the top row, taking into account the exact sequence

$$0 \longrightarrow \wedge^2 M_L \longrightarrow (pr_1)_*(pr_2^* L \otimes pr_3^* L \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,2} - \Delta_{1,3} - \Delta_{2,3})) \longrightarrow P_L \longrightarrow 0$$

used to construct P_L .

Finally, we need to show that the map $(H^0 \mathcal{I}_C(2)) \otimes \mathcal{O}_C \longrightarrow N_{C/\mathbf{P}}^\vee(2)$ is induced by the canonical map $\mathcal{I}_C \longrightarrow \mathcal{I}_C / \mathcal{I}_C^2 \cong N_{C/\mathbf{P}}^\vee$. This can be done by a direct calculation, or as follows:

Note that diagram (2) above can alternatively be obtained as pullback of a similar diagram that lives on the blow-up of $\mathbf{P}^r \times \mathbf{P}^r$ in the diagonal.

Now given an arbitrary quadric Q containing C , pulling the diagram back to the blow-up of $Q \times Q$ in its diagonal, and taking the appropriate direct images, we obtain the top map of the following diagram

$$(5) \quad \begin{array}{ccc} (H^0 \mathcal{I}_Q(2)) \otimes \mathcal{O}_Q & \xrightarrow{\sim} & N_{Q/\mathbf{P}}^\vee(2) = \mathcal{O}_Q \\ \downarrow & & \downarrow \\ (H^0 \mathcal{I}_C(2)) \otimes \mathcal{O}_C & \longrightarrow & N_{C/\mathbf{P}}^\vee(2). \end{array}$$

The commutativity of diagram (5) now provides the desired identification of the map. \square

3. CONCLUSION OF THE PROOF

Theorem 3.1. $H^0(\mathcal{I}_C(2)) \otimes H^0 B \longrightarrow H^0(N_{C/\mathbf{P}}^\vee(2) \otimes B)$ is surjective, if one of the following conditions holds:

- (i) $g = 0$, $d = \deg(C) \geq 1$, $\deg B \geq 1$;
- (ii) $g = 1$, $d \geq 4$, $\deg B \geq 3$;

- (iii) $g \geq 2$, $d = 2g + 2$, $\deg B \geq 2g + 3$;
- (iv) $g \geq 2$, $d \geq 2g + 3$, $\deg B \geq 2g + 1$.

Proof of (3.1). By (2.2) it suffices to show that $H^1(C, (pr_1)_*\mathcal{F}) = 0$ for $\mathcal{F} = pr_1^*B \otimes pr_2^*L \otimes pr_3^*L \otimes \mathcal{O}(-\Delta_{1,2} - \Delta_{1,3} - \Delta_{2,3})$ where $L = \mathcal{O}_C(1)$. If we knew that the higher direct images $R^1(pr_1)_*\mathcal{F}$ and $R^2(pr_1)_*\mathcal{F}$ both vanish, then Leray's spectral sequence implies that $H^1(pr_1)_*\mathcal{F} = H^1\mathcal{F}$, and the latter vanishes by (3.2) resp. (3.3) below.

To compute the higher direct images, we note that $R^1(pr_{1,2})_*\mathcal{F} = pr_1^*B \otimes pr_2^*L \otimes \mathcal{O}(-\Delta) \otimes R^1(pr_{1,2})_*(pr_3^*L \otimes \mathcal{O}(-\Delta_{1,3} - \Delta_{2,3})) = 0$, because L is non-special and very ample, hence $R^2(pr_1)_*\mathcal{F} = 0$, and $R^1(pr_1)_*\mathcal{F} = B \otimes R^1(pr_1)_*(M_{2,L} \otimes pr_2^*L \otimes \mathcal{O}(-\Delta))$.

Now consider the left vertical sequence of diagram (2) in the proof of (2.2). After tensoring with $pr_2^*L \otimes \mathcal{O}(-\Delta)$ and applying $(pr_1)_*$, we find that the stalks of $R^1(pr_1)_*(M_{2,L} \otimes pr_2^*L \otimes \mathcal{O}(-\Delta))$ sit in the exact sequence

$$\begin{aligned} M_{L,x} \otimes H^0L(-x) &\xrightarrow{f} H^0L^2(-2x) \longrightarrow R^1(pr_1)_*(M_{2,L} \otimes pr_2^*L \otimes \mathcal{O}(-\Delta))_x \\ &\longrightarrow M_{L,x} \otimes H^1L(-x). \end{aligned}$$

$M_{L,x}$ can be canonically identified with $H^0L(-x)$, and f with the multiplication map $H^0L(-x) \otimes H^0L(-x) \longrightarrow H^0L^2(-2x)$. The assumption on d implies that f is surjective and that the right term vanishes for every $x \in C$, therefore $R^1(pr_1)_*\mathcal{F} = 0$. \square

Proof of (1.4). The line bundle $B = \omega_C \otimes \mathcal{O}_C(\Gamma - 2)$ has degree $2g - 2 + n - 2d = n - 2r - 2$ which is at least $2g + 1$ (for $g \leq r - 3$) resp. $2g + 3$ (for $g = r - 2$), therefore the theorem follows from (2.1) and (3.1). \square

Proof of (1.5). $B = \omega_C \otimes \mathcal{O}_C(\Gamma - 2)$ is a general line bundle of degree $2g - 2 + n - 2d \geq g + 3$, and, going back to the proofs of (1.4) and (3.1), it suffices to show that $H^1(pr_1^*B \otimes pr_2^*\mathcal{O}_C(1) \otimes pr_3^*\mathcal{O}_C(1) \otimes \mathcal{O}(-\Delta_{1,2} - \Delta_{1,3} - \Delta_{2,3})) = 0$. But that result will be shown as (6) in the proof of (3.2) below. \square

Theorem 3.2. *Let C be a curve of genus g , let L_1 , L_2 and L_3 be line bundles on C with $\deg L_1 \geq 2g + 1$, $\deg L_2 \geq 2g + 2$, $\deg L_3 \geq 2g + 3$. Then*

$$H^1(pr_1^*L_1 \otimes pr_2^*L_2 \otimes pr_3^*L_3 \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,2} - \Delta_{1,3} - \Delta_{2,3})) = 0.$$

Proof. Choose a line bundle L'_3 of degree $g + 3$ with the following properties:

- (i) L'_3 is very ample and non-special,
- (ii) $H^1(L_1(p) \otimes L'^{-1}_3) = 0$ for general $p \in C$,
- (iii) $H^1(\wedge^2 M_{L'_3} \otimes L_1) = 0$,
- (iv) $H^1(L_2 \otimes L'^{-1}_3) = 0$.

Each of them separately holds on a Zariski-open subset of the Picard variety of line bundles of degree $g + 3$, as long as $\deg(L_1(p) \otimes L'^{-1}_3) \geq g - 1$ and $\deg(L_2 \otimes L'^{-1}_3) \geq g - 1$.

Since $\deg(L_3 \otimes L'_3{}^{-1}) \geq g$, we have $H^0(L_3 \otimes L'_3{}^{-1}) \neq 0$. If the divisor of the corresponding section consists of pairwise distinct points, we have a short exact sequence

$$0 \longrightarrow pr_3^* L'_3 \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,3} - \Delta_{2,3}) \longrightarrow pr_3^* L_3 \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,3} - \Delta_{2,3}) \longrightarrow \bigoplus_i \mathcal{O}_{x_3=p_i}(-p_i, -p_i) \longrightarrow 0;$$

in the general case, we will still have a short exact sequence starting with the two terms above. However, the term on the right may not split as a direct sum, but instead have a filtration with graded pieces $\mathcal{O}_{x_3=p_i}(-p_i, -p_i)$.

Now (3.2) will follow from

$$(6) \quad H^1(pr_1^* L_1 \otimes pr_2^* L_2 \otimes pr_3^* L'_3 \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,2} - \Delta_{1,3} - \Delta_{2,3})) = 0,$$

$$(7) \quad H^1(pr_1^* L_1(-p_i) \otimes pr_2^* L_2(-p_i) \otimes \mathcal{O}_{C \times C}(-\Delta)) = 0.$$

It is well-known that (7) holds, if $\deg L_1(-p_i) \geq 2g$ and $\deg L_2(-p_i) \geq 2g + 1$; i.e., if $\deg L_1 \geq 2g + 1$ and $\deg L_2 \geq 2g + 2$.

Since L'_3 is very ample and non-special, $H^1 L'_3(-x_1 - x_2) = 0$ for all $(x_1, x_2) \in C \times C$; hence $R^1(pr_{1,2})_*(pr_3^* L'_3 \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,3} - \Delta_{2,3})) = 0$. Denoting $M_{2,L'_3} = (pr_{1,2})_*(pr_3^* L'_3 \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,3} - \Delta_{2,3}))$, Leray's spectral sequence reduces (6) to

$$(8) \quad H^1(M_{2,L'_3} \otimes pr_1^* L_1 \otimes pr_2^* L_2 \otimes \mathcal{O}_{C \times C}(-\Delta)) = 0.$$

In the next step, apply $(pr_{1,2})_*$ to

$$0 \longrightarrow pr_3^* L'_3 \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,3} - \Delta_{2,3}) \longrightarrow pr_3^* L'_3 \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,3}) \longrightarrow pr_3^* L'_3 \otimes \mathcal{O}_{\Delta_{2,3}}(-\Delta_{1,3}) \longrightarrow 0$$

to obtain a sequence

$$0 \longrightarrow M_{2,L'_3} \longrightarrow pr_1^* M_{L'_3} \longrightarrow pr_2^* L'_3 \otimes \mathcal{O}_{C \times C}(-\Delta) \longrightarrow 0.$$

Its second exterior power is

$$0 \longrightarrow \wedge^2 M_{2,L'_3} \longrightarrow \wedge^2 pr_1^* M_{L'_3} \longrightarrow M_{2,L'_3} \otimes pr_2^* L'_3 \otimes \mathcal{O}_{C \times C}(-\Delta) \longrightarrow 0.$$

Since M_{2,L'_3} is a vector bundle of rank $H^0 L'_3(-x - y) = 2$, $\wedge^2 M_{2,L'_3}$ is a line bundle, we calculate

$$\wedge^2 M_{2,L'_3} = pr_1^* L'_3{}^{-1} \otimes pr_2^* L'_3{}^{-1} \otimes \mathcal{O}_{C \times C}(\Delta),$$

and (8) will now follow from

$$(9) \quad H^1(pr_1^*(\wedge^2 M_{L'_3} \otimes L_1) \otimes pr_2^*(L_2 \otimes L'_3{}^{-1})) = 0;$$

$$(10) \quad H^2(pr_1^*(L_1 \otimes L'_3{}^{-1}) \otimes pr_2^*(L_2 \otimes L'_3{}^{-2}) \otimes \mathcal{O}_{C \times C}(\Delta)) = 0.$$

Now (iii) and (iv) together imply (9), while (10) follows from Leray's spectral sequence, since (ii) above implies that $R^1(pr_2)_*(pr_1^*(L_1 \otimes L'_3{}^{-1}) \otimes \mathcal{O}_{C \times C}(\Delta))$ is supported in a (possibly empty) set of points. \square

Remark 3.3. For curves of genus $g \leq 1$, some of the arguments in the above proof can be improved:

For elliptic curves, any line bundle L'_3 of degree 3 is very ample and gives a plane embedding. Hence the vanishing result of (3.2) holds for line bundles L_1, L_2, L_3 , as soon as $\deg L_1 \geq 3$, $\deg L_2 \geq 4$ and $\deg L_3 \geq 4$. Therefore (1.4) even holds for $\alpha = 1$, $r = 3$, in line with Eisenbud and Harris' result.

For rational curves, we have the vanishing for $\deg L_1 \geq 1$, $\deg L_2 \geq 1$ and $\deg L_3 \geq 1$, hence (1.4) is also valid for $\alpha = 0$, $r = 2$ in line with Castelnuovo's result.

The bounds in (3.1) and (3.2) are sharp, as can be seen from the following

Example 3.4. Let C be a hyperelliptic curve of genus $g \geq 2$.

1. ($\deg L_1 = \deg L_2 = 2g + 1$, arbitrary L_3) Embed C by a line bundle L of degree $2g + 1$. Then $(H^0 \mathcal{I}_C(2)) \otimes \mathcal{O}_C \rightarrow N_C^\vee(2)$ is not surjective.
2. ($\deg L_1 = \deg L_2 = \deg L_3 = 2g + 2$) Embed C by a line bundle L of degree $2g + 2$. Then $H^0 \mathcal{I}_C(2) \otimes H^0 L \rightarrow H^0 N_C^\vee(3)$ is not surjective.

Proof. 1. Since C is hyperelliptic, its image in \mathbf{P}^{g+1} lies on a rational normal scroll S of degree g [4]. Any quadric not containing S will intersect S in a divisor of degree $2g$, hence cannot contain C .

2. According to the exact sequence

$$0 \rightarrow P_L \rightarrow (H^0 \mathcal{I}_C(2)) \otimes \mathcal{O}_C \rightarrow N_C^\vee(2) \rightarrow 0,$$

it suffices to show that $H^1(P_L \otimes L) \neq 0$. As $H^1(\wedge^2 M_L \otimes L) = 0$, the exact sequence

$$0 \rightarrow \wedge^2 M_L \rightarrow (pr_1)_*(pr_2^* L \otimes pr_3^* L \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,2} - \Delta_{1,3} - \Delta_{2,3})) \rightarrow P_L \rightarrow 0$$

on C and Leray's spectral sequence show that

$$H^1(P_L \otimes L) \cong H^1(pr_1^* L \otimes pr_2^* L \otimes pr_3^* L \otimes \mathcal{O}_{C \times C \times C}(-\Delta_{1,2} - \Delta_{1,3} - \Delta_{2,3})).$$

Filtering L as in the proof of (3.2), we find that this cohomology group surjects onto $H^1(C \times C, pr_1^* L(-p) \otimes pr_2^* L(-p) \otimes \mathcal{O}_{C \times C}(-\Delta))$ which is non-trivial for hyperelliptic C [8, 1.4.2]. \square

4. ON LINEAR NORMALITY, AND TWO APPLICATIONS

4.1. ABOUT THE ASSUMPTION OF LINEAR NORMALITY

Recall the setup in which we are working:

$X \subset \mathbf{P}^{r+1}$ is a nondegenerate reduced irreducible curve of degree n and genus $g > \pi_\alpha(n, r)$ for some $\alpha \leq r$. $C \subset \mathbf{P}^r$ is a curve of degree $\leq r - 1 + \alpha$ containing a general hyperplane section Γ of X .

Our results in sections 2 and 3 apply to linearly normal curves C whose ideal sheaf is generated by quadrics.

We would like to point out:

- (i) Even if X is linearly normal, C may not have this property; e.g., let C_0 be a curve of genus $g \geq 1$, let D be a line of degree $2g + k$, let $S = C_0 \times \mathbf{P}^1$, embedded into $\mathbf{P}^{2g+2k+1}$ by the line bundle $pr_1^*D \otimes \mathcal{O}_{\mathbf{P}^1}(1)$ as a surface of degree $4g + 2k$, and let X be the intersection of S with a hypersurface of large degree; here $\alpha = 2g + 1$ and $h^1\mathcal{I}_C(1) = h^2\mathcal{O}_S = g$.
- (ii) The intersection of the quadrics containing Γ may have dimension 2; let $k = 1$ and let C_0 be hyperelliptic in the previous example; the ideal sheaves of S and its hyperplane section C are generated by quadrics and cubics.

4.2. A CRITERION FOR A CURVE TO BE LOCALLY AN INTERSECTION OF QUADRICS

Let L be a very ample line bundle of degree $d \geq 2g + 2 - 2h^1(L) - \text{Cliff}(C)$ defining an embedding $C \subset \mathbf{P}(H^0(L)) = \mathbf{P}^r$. Green and Lazarsfeld have shown that C is scheme-theoretically cut out by quadrics unless it has a tri-secant line. Their proof is outlined in [8, 2.4.2].

We would like to offer an alternative proof, starting with

Proposition 4.1. *Let L be a very ample line bundle on a curve C which embeds C as a quadratically normal curve, $x \in C$. Then the following conditions are equivalent:*

- (i) *C is scheme-theoretically cut out by quadrics at x .*
- (ii) *The line bundle $L(-x)$ is very ample (i.e., C has no trisecant line through x) and embeds C as a quadratically normal curve.*

Proof. Consider the diagram (4) in the proof of (3.2). Note first that the two bottom rows of the diagram exist for an arbitrary variety and are exact. Quadratic normality implies the surjectivity of the vertical map in the middle (compare diagram (3)), and we identified the kernels of the vertical maps, hence the snake lemma yields an exact sequence

$$H^0\mathcal{I}_C(2) \otimes \mathcal{O}_C \longrightarrow N_{C/\mathbf{P}}^\vee(2) \longrightarrow \text{Coker}(S^2M_L \longrightarrow R_{L^2}) \longrightarrow 0.$$

Since the map $S^2M_{L,x} \longrightarrow R_{L^2,x}$ can be identified with $S^2H^0L(-x) \longrightarrow H^0L^2(-2x)$, this implies the equivalence of the two conditions. \square

Green and Lazarsfeld's result is now an immediate consequence of their earlier result that the embedding by a very ample line bundle of degree $d \geq 2g + 1 - 2h^1(L) - \text{Cliff}(C)$ is quadratically normal [6].

4.3. COHOMOLOGY OF THE SQUARE OF THE IDEAL SHEAF

Our results permit to answer a question, posed independently by A. Bertram and J. Wahl:

Proposition 4.2. *Let $C \subset \mathbf{P}^r$ be a curve of genus g , embedded by a complete linear system of degree $d \geq 2g + 3$. Then $H^1\mathcal{I}_C^2(3) = 0$.*

Proof. It suffices to show that $H^0 \mathcal{I}_C(3) \longrightarrow H^0 N_{C/\mathbf{P}}^\vee(3)$ is surjective. But by (3.1), the map $H^0 \mathcal{I}_C(2) \otimes H^0 \mathcal{O}_{\mathbf{P}^r}(1) \longrightarrow H^0 N_{C/\mathbf{P}}^\vee(3)$ is surjective, and it factors through $H^0 \mathcal{I}_C(3)$. \square

(3.4) shows that the vanishing fails for hyperelliptic curves embedded by a line bundle of degree $2g + 2$, hence our result is the best possible without further assumptions on C .

Wahl is particularly interested in the canonical embedding [11]. He shows that the vanishing of $H^1 \mathcal{I}_C^2(3)$ holds for the general canonical curve of genus $g \geq 3$, but fails for curves with $\text{Cliff}(C) \leq 2$.

The picture for canonical curves is almost completed by recent work of Arbarello, Bruno and Sernesi [1] who show that $H^1 \mathcal{I}_C^2(3)$ vanishes for all curves of genus $g \geq 8$ and $\text{Cliff}(C) \geq 3$.

A natural generalization of (4.2) in the spirit of [6] would be the following

Conjecture 4.3. *Let $C \subset \mathbf{P}^r$ be a curve of genus g , embedded by a complete linear system $|L|$ of degree d , and suppose that $h^1 L \leq 1$. If $d \geq 2g + 3 - 2h^1(L) - \text{Cliff}(C)$, then $H^1 \mathcal{I}_C^2(3) = 0$.*

REFERENCES

- [1] Enrico Arbarello, Andrea Bruno and Edoardo Sernesi. On two conjectures of J. Wahl. *arXiv preprint arXiv:1507.05000* 2, 2015
- [2] Enrico Arbarello, Maurizio Cornalba, Phillip Griffiths and Joe Harris, *Geometry of Algebraic Curves, I*, Grundlehren der mathematischen Wissenschaften, vol. 267, Springer-Verlag, New York, 1985.
- [3] David Eisenbud, Mark Green and Joe Harris. Higher Castelnuovo theory. *Astérisque* (218);187–202, 1993. Journées de Géométrie Algébrique d’Orsay (Orsay, 1992).
- [4] David Eisenbud and Joe Harris. On varieties of minimal degree (a centennial account). In *Algebraic geometry, Bowdoin*, (Brunswick, Maine, 1985), Proc. Sympos. Pure Math. vol. 46, p. 3-13, Amer. Math. Soc., Providence, RI, 1987.
- [5] G. Fano. Sopra le curve di dato ordine e dei massimi generi in uno spazio qualunque. *Mem. Acad. Sci. Torino*, 44 (1894), 335–382.
- [6] Mark Green and Robert Lazarsfeld, On the projective normality of complete linear series on an algebraic curve, *Invent. Math.* 83 (1985), 73–90.
- [7] Joe Harris (with the collaboration of David Eisenbud). *Curves in projective space*, Séminaire de Mathématiques Supérieures vol. 85, Presses de l’Université de Montréal, Que., 1982.
- [8] Robert Lazarsfeld. A sampling of vector bundle techniques in the study of linear series. In *Lectures on Riemann surfaces* (Trieste, 1987), p. 500–559. World Sci. Publ., Teaneck, NJ, 1989.
- [9] Ivan Petrakiev. Castelnuovo theory via Gröbner bases, *J. Reine Angew. Math.* 619 (2008), 49–73.

- [10] Miles Reid. Quadrics through a canonical surface. In *Algebraic geometry (L'Aquila, 1988)*, Lecture Notes in Mathematics vol. 1417, p. 191–213. Springer, Berlin, 1990.
- [11] Jonathan Wahl. On cohomology of the square of an ideal sheaf, *J. Algebraic Geom.* 6 (1987), 481–511.

FLURSTRASSE 49, 82110 GERMERING, GERMANY

E-mail address: Juergen.Rathmann@yahoo.com